

Constrained proportional integral control of dynamical distribution networks with state constraints*

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Abstract—This paper studies a basic model of a dynamical distribution network, where the network topology is given by a directed graph with storage variables corresponding to the vertices and flow inputs corresponding to the edges. We aim at regulating the system to consensus, while the storage variables remain greater or equal than a given lower bound. The problem is solved by using a distributed PI controller structure with constraints which vary in time. It is shown how the constraints can be obtained by solving an optimization problem.

I. INTRODUCTION

In this paper we continue our study of the dynamics of distribution networks. Identifying the network with a directed graph we associate with every vertex of the graph a state variable corresponding to *storage*, and with every edge a control input variable corresponding to *flow*, possibly subject to constraints. In previous work [?], [?], [?] it has been shown under which conditions a constrained proportional-integral (PI) controller will regulate the system towards consensus in the presence of unknown constant external disturbances (corresponding to constant inflows and outflows of the network).

In quite a few cases of practical interest it is natural to require that the state variables of the distribution network will remain larger than a given minimal value, e.g. 0. A hydraulic network with state variables being the storage of fluid is a clear example of such a situation. On the other hand, the previously developed PI-controller can give rise to damped oscillatory behavior which may be violating such state constraints. The aim of the current paper is to modify the PI-controller in such a way that the lower bounds for the state variables will remain satisfied for all time while the system will still converge to consensus. This is done by adapting the constraints of the PI controller depending on the current flow values and information about state variables reaching their lower bound.

The structure of the paper is as follows. Preliminaries and notations are given in Section II. In Section III we briefly recall from our previous work [?], [?], [?] necessary and sufficient graphical conditions in order that a distributed PI controller structure, associating with every edge of the graph a controller state, will solve the regulation problem. In the absence of constraints the closed-loop system can

be identified as a port-Hamiltonian system, in line with the general definition of port-Hamiltonian system on graphs [?], [?], [?], [?]; see also [?], [?].

In Section IV we formulate the main problem of this paper, namely the adaptation of the constraints of the PI-controller such that the system will reach consensus while the state variables will remain greater or equal than a given lower bound. In Section V an optimal control protocol for the adaptation of the flow (control) constraints is developed, while the complete proof of the validity of the scheme is given in Section VI. The conclusions are contained in Section VII.

Finally we mention the following related work. In [?] an alternative scheme is given in order that the state variables remain nonnegative. However, this scheme does not respect mass conservation. In [?], the authors consider a similar distribution network model but with a proportional (P) controller instead of a PI controller. A discontinuous Lyapunov-based controller is given to stabilize the system without violating the storage and flow constraints; however without the advantages of a PI controller (i.e., robustness with respect to constant disturbances). In [?], using the same model as in [?], the authors focus on a different problem of driving the state to a small neighborhood of the reference value and relate the control input value at equilibrium to an optimization problem.

II. PRELIMINARIES AND NOTATIONS

We recall some standard definitions regarding directed graphs, as can be found e.g. in [?]. A *directed graph* \mathcal{G} consists of a finite set \mathcal{V} of *vertices* and a finite set \mathcal{E} of *edges*, together with a mapping from \mathcal{E} to the set of ordered pairs of \mathcal{V} , where no self-loops are allowed. Thus to any edge $e \in \mathcal{E}$ there corresponds an ordered pair $(v, w) \in \mathcal{V} \times \mathcal{V}$ (with $v \neq w$), representing the tail vertex v and the head vertex w of this edge.

A directed graph is specified by its *incidence matrix* B , which is an $n \times m$ matrix, n being the number of vertices and m being the number of edges, with $(i, j)^{\text{th}}$ element equal to 1 if the j^{th} edge is towards vertex i , and equal to -1 if the j^{th} edge is originating from vertex i , and 0 otherwise. Since we will only consider directed graphs in this paper ‘graph’ will throughout mean ‘directed graph’ in the sequel. A directed graph is *strongly connected* if it is possible to reach any vertex starting from any other vertex by traversing edges following their directions. A directed graph is called *weakly connected* if it is possible to reach any vertex from every other vertex using the edges *not* taking

*The work of the first author is supported by the Chinese Science Council (CSC). The research of the second author leading to these results has received support from the EU 7th Framework Programme [FP7/2007-2013] under grant agreement no. 257462 HYCON2 Network of Excellence.

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into account their direction. A graph is weakly connected if and only if $\ker B^T = \text{span } \mathbf{1}_n$. Here $\mathbf{1}_n$ denotes the n -dimensional vector with all elements equal to 1. A graph that is not weakly connected falls apart into a number of weakly connected subgraphs, called the connected components. The number of connected components is equal to $\dim \ker B^T$. For each vertex, the number of incoming edges is called the *in-degree* of the vertex and the number of outgoing edges its out-degree. A graph is called *balanced* if and only if the in-degree and out-degree of every vertex are equal. A graph is balanced if and only if $\mathbf{1}_n \in \ker B$.

Given a graph, we define its *vertex space* as the vector space of all functions from \mathcal{V} to some linear space \mathcal{R} . In the rest of this paper we will take $\mathcal{R} = \mathbb{R}$, in which case the vertex space can be identified with \mathbb{R}^n . Similarly, we define its *edge space* as the vector space of all functions from \mathcal{E} to $\mathcal{R} = \mathbb{R}$, which can be identified with \mathbb{R}^m . In this way, the incidence matrix B of the graph can be also regarded as the matrix representation of a linear map from the edge space \mathbb{R}^m to the vertex space \mathbb{R}^n .

Notation: For $a, b \in \{\mathbb{R}, \pm\infty\}^m$ the notation $a \leq b$ (resp. $a < b$) will denote element-wise inequality $a_i \leq b_i$ (resp. $a_i < b_i$), $i = 1, \dots, m$. For $a < b$ the multidimensional saturation function $\text{sat}(x; a, b) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is defined as

$$\text{sat}(x; a, b)_i = \begin{cases} a_i & \text{if } x_i \leq a_i, \\ x_i & \text{if } a_i < x_i < b_i, \\ b_i & \text{if } x_i \geq b_i, \end{cases} \quad i = 1, \dots, m. \quad (1)$$

III. A DYNAMICAL NETWORK MODEL WITH PI CONTROLLER

Consider the following dynamical system defined on the graph \mathcal{G}

$$\begin{aligned} \dot{x} &= Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\ y &= B^T \frac{\partial H}{\partial x}(x), \quad y \in \mathbb{R}^m, \end{aligned} \quad (2)$$

where $H : \mathbb{R}^n \rightarrow \mathbb{R}$ is a differentiable function, and $\frac{\partial H}{\partial x}(x)$ denotes the column vector of partial derivatives of H . Here the i -th element x_i of the state vector x is the state variable associated to the i -th vertex, while u_j is a flow input variable associated to the j -th edge of the graph. System (2) defines a *port-Hamiltonian system* ([?], [?]), satisfying the energy-balance

$$\frac{d}{dt} H = u^T y. \quad (3)$$

Note that geometrically its state space is the vertex space, its input space is the edge space, while its output space is the dual of the edge space [?]. Many distribution networks are of this form; see [?], [?] for further background.

Furthermore, we extend the dynamical system (2) with a vector d of *inflows and outflows*

$$\begin{aligned} \dot{x} &= Bu + Ed, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, \quad d \in \mathbb{R}^k \\ y &= B^T \frac{\partial H}{\partial x}(x), \quad y \in \mathbb{R}^m, \end{aligned} \quad (4)$$

where E is an $n \times k$ matrix whose columns consist of exactly one entry equal to 1 (inflow) or -1 (outflow), while

the rest of the elements is zero. Thus E specifies the k terminal vertices where flows can enter or leave the network (*sources* and *sinks*). Here d is regarded as a vector of *constant disturbances*

In [?], [?], [?] we investigated control schemes which ensure asymptotic load balancing of the state vector x irrespective of the unknown value of d , namely a proportional-integral output feedback (as in [?])

$$\begin{aligned} \dot{x}_c &= y, \\ \mu &= Ry + \frac{\partial H_c}{\partial x_c}(x_c), \end{aligned} \quad (5)$$

where R is a diagonal matrix with strictly positive diagonal elements r_1, \dots, r_m , μ and $H_c(x_c)$ denotes the output and storage function (energy) of the controller respectively. Note that this defines a *decentralized* control scheme if H is of the form $H(x) = H_1(x_1) + \dots + H_n(x_n)$, in which case the i -th input is given as r_i times the difference of the component of $\frac{\partial H}{\partial x}(x)$ corresponding to the head vertex of the i -th edge and the component of $\frac{\partial H}{\partial x}(x)$ corresponding to its tail vertex.

The j -th element of the controller state x_c can be regarded as an additional state variable corresponding to the j -th edge. Thus $x_c \in \mathbb{R}^m$, the edge space of the network. When $u(t) = -\mu(t)$, the closed-loop system resulting from the PI control (5) is given as

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} -BRB^T & -B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x}(x) \\ \frac{\partial H_c}{\partial x_c}(x_c) \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} d, \quad (6)$$

This is again a port-Hamiltonian system, with total Hamiltonian

$$H_{\text{tot}}(x, x_c) := H(x) + H_c(x_c),$$

and satisfying the energy-balance

$$\frac{d}{dt} H_{\text{tot}} = -\frac{\partial^T H}{\partial x}(x) B R B^T \frac{\partial H}{\partial x}(x) + \frac{\partial^T H}{\partial x}(x) E d \quad (7)$$

Consider now a constant disturbance \bar{d} for which there exists a *matching* controller state \bar{x}_c , i.e.,

$$E\bar{d} = B \frac{\partial H_c}{\partial x_c}(\bar{x}_c). \quad (8)$$

By modifying the total Hamiltonian $H_{\text{tot}}(x, x_c)$ into the candidate Lyapunov function

$$\begin{aligned} V_{\bar{d}}(x, x_c) &:= H(x) + H_c(x_c) \\ &\quad - \frac{\partial^T H_c}{\partial x_c}(\bar{x}_c)(x_c - \bar{x}_c) - H_c(\bar{x}_c), \end{aligned} \quad (9)$$

the following theorem is obtained [?], [?].

Theorem 1: Consider the system (4) on the graph \mathcal{G} in closed-loop with the PI-controller (5) with $u(t) = -\mu(t)$. Let the constant disturbance \bar{d} be such that there exists a \bar{x}_c satisfying the matching equation (8). Assume that $V_{\bar{d}}(x, x_c)$ is radially unbounded. Then the trajectories of the closed-loop system (6) will converge to an element of the load balancing set

$$\mathcal{E}_{\text{tot}} = \{(x, x_c) \mid \frac{\partial H}{\partial x}(x) = \alpha \mathbf{1}, \alpha \in \mathbb{R}, B \frac{\partial H_c}{\partial x_c}(x_c) = E\bar{d}\}. \quad (10)$$

if and only if \mathcal{G} is weakly connected.

Corollary 2: If $\ker B = 0$, which is equivalent [?] to the graph having no cycles, then for every \bar{d} there exists a unique \bar{x}_c satisfying (8), and convergence is towards the set $\mathcal{E}_{\text{tot}} = \{(x, \bar{x}_c) \mid \frac{\partial H}{\partial x}(x) = \alpha \mathbf{1}, \alpha \in \mathbb{R}, x_c = \bar{x}_c\}$.

Corollary 3: In case of the standard quadratic Hamiltonians $H(x) = \frac{1}{2}\|x\|^2$, $H_c(x_c) = \frac{1}{2}\|x_c\|^2$ there exists for every \bar{d} a controller state \bar{x}_c such that (8) holds if and only if

$$\text{im } E \subset \text{im } B. \quad (11)$$

Furthermore, in this case $V_{\bar{d}}$ equals the radially unbounded function $\frac{1}{2}\|x\|^2 + \frac{1}{2}\|x_c - \bar{x}_c\|^2$, while convergence will be towards the load balancing set $\mathcal{E}_{\text{tot}} = \{(x, x_c) \mid x = \alpha \mathbf{1}, \alpha \in \mathbb{R}, Bx_c = E\bar{d}\}$.

A necessary (and in case the graph is weakly connected necessary and sufficient) condition for the inclusion $\text{im } E \subset \text{im } B$ is that $\mathbf{1}^T E = 0$. In its turn $\mathbf{1}^T E = 0$ is equivalent to the fact that for every \bar{d} the total inflow into the network equals to the total outflow. The condition $\mathbf{1}^T E = 0$ also implies

$$\mathbf{1}^T \dot{x} = -\mathbf{1}^T B R B^T \frac{\partial H}{\partial x}(x) + \mathbf{1}^T E \bar{d} = 0, \quad (12)$$

implying (as in the case $d = 0$) that $\mathbf{1}^T x$ is a *conserved quantity* for the closed-loop system (6). In particular it follows that the limit value $\lim_{t \rightarrow \infty} x(t) \in \text{span}\{\mathbf{1}\}$ is determined by the initial condition $x(0)$.

In many cases of interest, the flows in the edges are constrained, namely $u(t) = \text{sat}(-\mu(t); u^-, u^+)$ where $u^-, u^+ \in \mathbb{R}^m$ are the lower and upper bounds for the flow constraints. It is shown in [?] how by adjusting the orientation of the graph we can assume without loss of generality that the flow constraints are *compatible with the orientation* in the sense that $u^+ \geq u^- \geq 0$. Furthermore, in [?] it is shown how for the disturbance satisfying the constrained version of the matching condition, i.e.,

$$E\bar{d} = B \text{sat}\left(\frac{\partial H_c}{\partial x_c}(\bar{x}_c); u^-, u^+\right), \quad (13)$$

the disturbance can be incorporated into the saturation bounds. It follows that, without loss of generality, we can focus on the closed-loop system without disturbance

$$\begin{aligned} \dot{x} &= B \text{sat}\left(-B^T \frac{\partial H}{\partial x}(x) - \frac{\partial H_c}{\partial x_c}(x_c), u^-, u^+\right), \\ \dot{x}_c &= B^T \frac{\partial H}{\partial x}(x). \end{aligned} \quad (14)$$

where for simplicity we have used an identity gain matrix $R = I$. The main results about system (14) are summarized in

Theorem 4: ([?]) Consider a network \mathcal{G} with dynamics (14) with compatible flow constraints. Then for any $u^- < u^+ \in \mathbb{R}_+^m$ such that $\cap_{i=1}^m [u_i^-, u_i^+]$ contains an open interval, the trajectories of (14) converge to

$$\begin{aligned} \mathcal{E}_{\text{tot}} = \{ & (x, x_c) \mid \frac{\partial H}{\partial x}(x) = \alpha \mathbf{1}, \alpha \in \mathbb{R}, \\ & B \text{sat}(-x_c, u^-, u^+) = 0 \} \end{aligned} \quad (15)$$

if and only if the graph in the (only) orientation compatible with the flow constraints is *strongly connected* and *balanced*.

For any network with given orientation and constraints on the edges, we can define the *interior point condition*.

Definition 5: (Interior Point Condition, [?]) Given a directed graph with arbitrary constraints $[u^-, u^+]$ (perhaps not compatible with the orientation), the network will be said to satisfy the interior point condition if there exists a vector $z \in [u^-, u^+]$ such that

$$B \text{sat}(z; u^-, u^+) = Bz = 0, \quad (16)$$

and the set of edges along which the corresponding element of z is an interior point of the constraint interval contains a spanning tree.

Theorem 6: ([?]) Consider the dynamical system (14) defined on a weakly connected directed graph with compatible constraints $[u^-, u^+]$. Then the trajectories will converge to

$$\mathcal{E}_{\text{tot}} = \{(x, x_c) \mid \frac{\partial H}{\partial x}(x) = \alpha \mathbf{1}_n, B \text{sat}(-x_c; u^-, u^+) = 0\}. \quad (17)$$

if and only if the network satisfies the interior point condition.

IV. CONSTRAINED PI-CONTROLLERS MAINTAINING A LOWER BOUND FOR THE STATE VARIABLES

Although, as summarized in the previous section, the PI controller (5) under general conditions is successful in obtaining consensus (load balancing) for a system with flow constraints and (unknown) inflows and outflows, it introduces damped oscillatory behavior which may result in solution trajectories in which (part of the) state variables x_i become negative. For certain applications this may be undesirable or infeasible, as illustrated by the following example.

Example 4.1 (Hydraulic network): Consider a hydraulic network, modeled as a directed graph with vertices (nodes) corresponding to reservoirs, and edges (branches) corresponding to pipes. Let x_i be the volume of fluid stored at vertex i , and u_j the flow through edge j . Then the mass balance of the network is summarized as (2). Let $H(x)$ denote the stored energy in the reservoirs (e.g., gravitational energy). For cylindric reservoirs, $x_i = S_i h_i$, $H_i = \frac{1}{2} \rho S_i g h_i^2 = \frac{\rho g}{2 S_i} x_i^2$ where S_i is the bottom area, h_i is the height of liquid of i th reservoir respectively, and g is gravity coefficient. Then $P_i := \frac{\partial H}{\partial x_i}(x) = \rho g h_i = \frac{\rho g x_i}{S_i}$, $i = 1, \dots, n$, are the *pressures* at the vertices, and the output vector $y = B^T \frac{\partial H}{\partial x}(x)$ is the vector whose j^{th} element is the pressure *difference* $P_i - P_k$ across the j^{th} edge linking vertex k to vertex i . The proportional part $u = -Ry$ of the PI controller (5) corresponds to adding *damping* to the dynamics (proportional to the pressure differences along the edges). The integral part of the controller has the interpretation of adding *compressibility* to the compressible fluid network dynamics. Using this emulated compressibility, the PI-controllers (5), both in the unconstrained and constrained case, are able to regulate the fluid network to a load balancing situation where all pressures P_i are equal, irrespective of the constant inflow and outflow \bar{d} satisfying the matching condition (8) and (13) respectively.

On the other hand, depending on the initial conditions and the flow constraints (or the value of the constant inflows and outflows) some of the state variables x_i may become negative, which is clearly infeasible. This motivates to include an additional requirement to the control problem in this example; namely to control the system in such a way that the fluid volume x_i in all reservoirs remains greater or equal than a given positive value for all future time, while the vector of pressures $\frac{\partial H}{\partial x_i}$ still converges to consensus.

Remark 7: Note that in the *unconstrained* case and with $H(x) = \frac{1}{2}\|x\|^2$ the *proportional* controller $u = -Ry$ with R a positive diagonal matrix, has always the property that the evolution of the closed-loop system

$$\dot{x} = -BRB^T x$$

remains in the positive orthant \mathbb{R}_+^n . This directly follows from the properties of the weighted Laplacian matrix BRB^T : whenever at a certain moment $x_i(t) = 0$ then $\dot{x}_i(t) \geq 0$. On the other hand, the proportional controller does not share the favorable properties of the proportional integral controller; in particular, its robustness with respect to unknown constant disturbances.

In order to keep the vector of state variables x of the system greater than or equal to a given vector¹ $\gamma \in \mathbb{R}^n$ for all future times, i.e. $x(t) \geq \gamma, \forall t \geq 0$, while driving the dual variables $\frac{\partial H}{\partial x}(x(t))$ to consensus, we develop in this paper a *modified version of the constrained PI-controller*, which is based on adjusting the flow constraints depending on information regarding the values of the state variables $x_i, i = 1, \dots, n$. More precisely, we develop a hybrid constrained PI controller, which keeps track of the state variables x_i reaching the given lower bound γ_i , and accordingly adapts the flow constraints; i.e., the allowed values of the PI-controller.

For clarity of exposition we focus in this paper on systems without in/outflows (i.e., $d = 0$). More precisely we consider throughout the following system

$$\begin{aligned} \dot{x} &= B \text{sat}(-B^T \frac{\partial H}{\partial x}(x) - \frac{\partial H_c}{\partial x_c}(x_c), u^-(t), u^+(t)), \\ \dot{x}_c &= B^T \frac{\partial H}{\partial x}(x). \end{aligned} \quad (18)$$

where $H_c(x_c) \in \mathbb{C}^1$, $H(x) = \sum_{i=1}^n H_i(x_i) \in \mathbb{C}^2$ with $H_i(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ be strictly convex and $\arg \min H_i(x_i) = \gamma_i, i = 1, 2, \dots, n$, $u^-(t)$ and $u^+(t)$ are parameters to be designed.

Example 4.2 (Hydraulic network continued): Suppose we want to keep the fluid storage of each reservoir greater or equal than a given value $S_i \gamma \geq 0$, then we can *modify* the Hamiltonian as $H_i(x_i) = \frac{\rho g}{2S_i}(x_i - S_i \gamma)^2$, which is the *extra* stored energy in the i th reservoir compared to the stored energy at height γ . In this case $\frac{\partial H}{\partial x_i} = \rho g(h_i - \gamma)$ which is the increase of pressure with regard to the pressure at height γ .

¹Recall that the inequality of vectors is defined as element-wise inequality.

V. THE DESIGN OF THE FLOW CONSTRAINTS

The only situation in which a state variable x_i can become smaller than the given lower bound γ_i is that at a certain time instant t , $x_i(t) = \gamma_i$ and $\dot{x}_i(t) < 0$. The basic idea underlying the design of the time-varying flow constraints is to eliminate this situation by adding saturation on the flows in the edges in such a way that $\dot{x}_i(t) \geq 0$ whenever $x_i(t) = \gamma_i$. For each time t and each vertex v_i , the edges adjacent to it can be divided into two sets

$$\begin{aligned} f_{v_i}^{\text{in}}(t) &= \{e_j \in \mathcal{E} \mid B_{ij}\mu_j > 0\} \\ f_{v_i}^{\text{out}}(t) &= \{e_j \in \mathcal{E} \mid B_{ij}\mu_j < 0\} \end{aligned} \quad (19)$$

where μ_j is the output of the controller on the edge e_j . For each time t , the vertices of the network can be divided into the following subsets, referred to as *white*, *gray* and *black* (with the last category divided into two subsets)

$$\begin{aligned} \mathcal{V}^W(t) &= \{v_i \in \mathcal{V} \mid x_i(t) > \gamma_i\} \\ \mathcal{V}^G(t) &= \{v_i \in \mathcal{V} \mid x_i(t) = \gamma_i\} \\ \mathcal{V}^{B1}(t) &= \{v_i \in \mathcal{V}^G \mid B(i, :)\mu(t) < 0\} \\ \mathcal{V}^{B2}(t) &= \{v_i \in \mathcal{V}^G \mid \exists v_j \in \mathcal{V}^{B1} \text{ s.t. } f_{v_i}^{\text{in}}(t) \cap f_{v_j}^{\text{out}}(t) \neq \emptyset\} \end{aligned} \quad (20)$$

where $B(i, :)$ is the i th row of B . Furthermore, we denote $\mathcal{V}^B(t) = \mathcal{V}^{B1}(t) \cup \mathcal{V}^{B2}(t)$.

Example 5.1: Let us consider a part of the network given as given in Fig.1, where $\mu_i(t), i = 1, 2, 3$ are the outputs of the controllers on the corresponding edges. This example shows that by setting $u(t) = -\mu(t)$, the states of *black* nodes can become negative. Indeed suppose that at time t the state variable at v_2 , i.e. $x_2(t)$, decreases to γ_2 , while $-\mu_2(t) - \mu_3(t) > -\mu_1(t) \geq 0$, then $\dot{x}_2(t) < 0$.

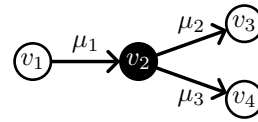


Fig. 1. Explanation about how the *black* vertices may exhibit negative state values, with μ_i the output of the controller on the i -th edge.

Let us denote the set of outgoing edges of all vertices in $\mathcal{V}^B(t)$, i.e., $\cup_{v_i \in \mathcal{V}^B(t)} f_{v_i}^{\text{out}}$, as $\mathcal{E}_{\text{out}}^B(t)$. Along the edges in $\mathcal{E} \setminus \mathcal{E}_{\text{out}}^B(t)$, we set $u(t) = -\mu(t)$, while along the edges in $\mathcal{E}_{\text{out}}^B(t)$, a saturation is imposed defined as $u(t) = \text{sat}(-\mu(t), -|\eta^*(t)|, |\eta^*(t)|)$, where $\eta^*(t)$ is the optimal solution of the following optimization problem

$$\begin{aligned} \min_{\eta} \quad & \sum_{e_j \in \mathcal{E}_{\text{out}}^B(t)} \frac{1}{2\mu_j(t)} ((\eta_j - \mu_j(t))^2 + \eta_j^2) \\ \text{s.t.} \quad & B(i, :)\eta = 0, \forall v_i \in \mathcal{V}^B(t), \\ & \eta_j = -\mu_j(t), \text{ if } e_j \in \mathcal{E} \setminus \mathcal{E}_{\text{out}}^B(t). \end{aligned} \quad (21)$$

where η has the same dimension as $\mathcal{E}_{\text{out}}^B(t)$.

More precisely, let us denote

$$\eta_j^+(t) = \begin{cases} |\eta_j^*(t)| & \text{if } e_j \in \mathcal{E}_{\text{out}}^B(t) \\ +\infty & \text{else,} \end{cases} \quad j = 1, \dots, m. \quad (22)$$

then the closed-loop can be written as

$$\begin{aligned}\dot{x} &= B \text{sat}(-B^T \frac{\partial H}{\partial x}(x) - \frac{\partial H_c}{\partial x_c}(x_c), -\eta^+(t), \eta^+(t)), \\ \dot{x}_c &= B^T \frac{\partial H}{\partial x}(x).\end{aligned}\quad (23)$$

Example 5.2: Continuing Example 5.1, suppose at time t , the flows are subject to $-\mu_2(t) - \mu_3(t) > -\mu_1(t) \geq 0$ and $x_2(t) = \gamma_2$. Then the above control protocol will set $u_1(t) = -\mu_1(t)$, $\eta_2^+(t) = |\frac{\mu_1(t)\mu_2(t)}{\mu_2(t)+\mu_3(t)}|$, $\eta_3^+(t) = |\frac{\mu_1(t)\mu_3(t)}{\mu_2(t)+\mu_3(t)}|$ and $u_2(t) = \text{sat}(-\mu_2(t), -\eta_2^+(t), \eta_2^+(t))$, $u_3(t) = \text{sat}(-\mu_3(t), -\eta_3^+(t), \eta_3^+(t))$ respectively.

Furthermore, the solution of the optimization problem (21) can be seen as the limit of the following algorithm.

Algorithm: *Initialization:* at time t when there are gray nodes in the network, set the initial value $\eta^0 = \mu(t) \in \mathbb{R}^m$. *Step k :* Let η^{k-1} be the value from the previous step $k-1$. Check if there exists a node i such that $B(i, :)\eta^{k-1} < 0$, then

$$\eta_j^k = \begin{cases} \frac{\sum_{e_j \in f_{v_i}^{\text{in}}} |\eta_j^{k-1}|}{\sum_{e_j \in f_{v_i}^{\text{out}}} |\eta_j^{k-1}|} \eta_j^{k-1} & \text{if } e_j \in f_{v_i}^{\text{out}} \\ \eta_j^{k-1} & \text{else,} \end{cases} \quad j = 1, \dots, m. \quad (24)$$

This algorithm is converging since in every iteration the absolute values of flows are non-increasing. It can be proved that $\lim_{k \rightarrow \infty} \eta^k(t) = \eta^*(t)$.

Example 5.3: In this example, we consider the structure of the network as given in Fig. 2. Suppose at time t , $x_1(t) > \gamma_1$, $x_2(t) = \gamma_2$, $x_3(t) = \gamma_3$ and the output of controller is $\mu(t) = [1, 3, 1, 2]^T$, then $\mathcal{V}^B(t) = \{v_2, v_3\}$. By using the algorithm, the flows on $f_2^{\text{out}}(t) \cup f_3^{\text{out}}(t) = \{e_2, e_3, e_4\}$, are saturated to the values $\frac{3}{2}, \frac{1}{2}$ and 1, respectively. Furthermore, it can be verified that $[\frac{3}{2}, \frac{1}{2}, 1]^T$ is the solution of optimization problem (21).

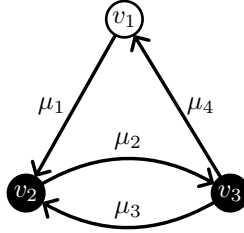


Fig. 2. Network of Example 5.3

VI. STABILITY ANALYSIS

In this section we will prove the stability of the system (23), and its convergence to consensus.

Since the right-hand-side of the system (23) is discontinuous, we will consider Filippov solutions. The notations are taken from [?].

Definition 8: ([?]) Let $\mathfrak{B}(\mathbb{R}^d)$ denote the collection of subsets of \mathbb{R}^d . For $X : \mathbb{R}^d \rightarrow \mathbb{R}^d$, define the *Filippov set-valued map* $F[X] : \mathbb{R}^d \rightarrow \mathfrak{B}(\mathbb{R}^d)$ as

$$F[X](x) \triangleq \bigcap_{\delta > 0} \bigcap_{\mu(S)=0} \overline{\text{co}}\{X(B(x, \delta) \setminus S)\} \quad (25)$$

Definition 9: A Filippov solution of $\dot{x}(t) = X(x(t))$ on $[0, t_1] \subset \mathbb{R}$ is an absolutely continuous map $x : [0, t_1] \rightarrow \mathbb{R}^d$ that satisfies

$$\dot{x}(t) \in F[X](x) \quad (26)$$

for almost all $t \in [0, t_1]$.

Here are two useful facts about computing the Filippov set-valued map.

Proposition 10: ([?]) *Product Rule:* If $X_1 : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $X_2 : \mathbb{R}^d \rightarrow \mathbb{R}^n$ are locally bounded at $x \in \mathbb{R}^d$, then

$$F[(X_1, X_2)^T](x) \subseteq F[X_1](x) \times F[X_2](x). \quad (27)$$

Moreover, if either X_1 or X_2 is continuous at x , then equality holds.

Matrix Transformation Rule: If $X : \mathbb{R}^d \rightarrow \mathbb{R}^m$ is locally bounded at $x \in \mathbb{R}^d$ and $Z : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ is continuous at $x \in \mathbb{R}^d$, then

$$F[ZX](x) = Z(x)F[X](x). \quad (28)$$

Theorem 11: Consider the system (23) on the graph \mathcal{G} in closed loop with the saturation bounds as given in (22). Assume that $H = \sum_{i=1}^n H_i(x_i) \in \mathbb{C}^2$ and $H_c \in \mathbb{C}^1$ are positive definitive, convex and radially unbounded with $\arg \min_{x \in \mathbb{R}^n} H(x) = \gamma \in \mathbb{R}^n$, where γ is a given vector. Then

- (i) $x(t) \geq \gamma$ for all $t > 0$ if $x(0) \geq \gamma$;
- (ii) the trajectories of the closed-loop system (23) will converge to an element of the load balancing set

$$\mathcal{E}_{\text{tot}} = \{(x, x_c) \mid \frac{\partial H}{\partial x}(x) = \alpha \mathbf{1}, \alpha \in \mathbb{R}^+, B \frac{\partial H_c}{\partial x_c}(x_c) = 0\}. \quad (29)$$

if and only if \mathcal{G} is weakly connected.

Proof: (i) It can be verified from the form of optimization problem (21) which grantee that $\dot{x}_i(t) \geq 0$ when $x_i(t) = \gamma_i$.

(ii) *Sufficiency.* First by using Proposition 10, the differential equations (23) are replaced by the differential inclusion

$$\begin{aligned}\begin{bmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{bmatrix} &\in F \left[\begin{bmatrix} B \text{sat}(-\mu(t), -\eta^+(t), \eta^+(t)) \\ B^T \frac{\partial H}{\partial x}(x) \end{bmatrix} \right] \\ &= \left[BF \left[\text{sat}(-\mu(t), -\eta^+(t), \eta^+(t)) \right] \right] \\ &\quad \left[B^T \frac{\partial H}{\partial x}(x) \right] \\ &\triangleq F(x, x_c)\end{aligned} \quad (30)$$

where the equality is implied by Proposition 10. Notice that the set-valued map $F(x, x_c)$ is locally bounded and its values are nonempty, compact and convex sets. Furthermore, for each $t \in \mathbb{R}$, $(x, x_c) \rightarrow F(x, x_c)$ is upper semi-continuous.

Take as Lyapunov function the Hamiltonian function

$$V(x, x_c) := H(x) + H_c(x_c), \quad (31)$$

which is differentiable. Then the set-valued Lie derivative $\mathcal{L}_F V : \mathbb{R}^{n+m} \rightarrow \mathfrak{B}(\mathbb{R})$ of V with respect to F at (x, x_c) is

defined as

$$\begin{aligned}\tilde{\mathcal{L}}_F V &= \{(\nabla V)^T \nu \mid \nu \in F(x, x_c)\} \\ &= \frac{\partial^T H}{\partial x}(x) B F \left[\text{sat}(-\mu(t), -\eta^+(t), \eta^+(t)) \right] \\ &\quad + \frac{\partial^T H}{\partial x}(x) B \frac{\partial H_c}{\partial x_c}(x_c)\end{aligned}\quad (32)$$

For the i -th edge, the Filippov set-valued map is given as

$$\begin{aligned}&F \left[\text{sat}(-\mu_j(t), -\eta_j^+(t), \eta_j^+(t)) \right] \\ &\subset \begin{cases} [0, \mu_i(t)] & e_i \in \mathcal{E}_{out}^B(t) \wedge \mu_i(t) > 0, \\ [\mu_i(t), 0] & e_i \in \mathcal{E}_{out}^B(t) \wedge \mu_i(t) < 0, \\ \{\mu_i(t)\} & \text{else,} \end{cases} \quad i = 1, \dots, m.\end{aligned}\quad (33)$$

For the i -th edge of \mathcal{G} on which $\eta_i^+(t) = +\infty$, i.e. $e_i \in \mathcal{E} \setminus \mathcal{E}_{out}^B(t)$, we have

$$\begin{aligned}&\frac{\partial^T H}{\partial x}(x(t)) B_i F \left[\text{sat}(-\mu_i(t), -\eta_i^+(t), \eta_i^+(t)) \right] \\ &\quad + \frac{\partial^T H}{\partial x}(x(t)) B_i \frac{\partial H_{c_i}}{\partial x_c}(x_c(t)) \\ &= - \frac{\partial^T H}{\partial x}(x(t)) B_i B_i^T \frac{\partial H}{\partial x}(x(t))\end{aligned}\quad (34)$$

where B_i is the i -th column of B .

For the i -th edge on which $\eta_i^+(t) < +\infty$, i.e. $e_i \in \mathcal{E}_{out}^B(t)$, we have that $\forall \nu \in F \left[\text{sat}(-\mu_i(t), -\eta_i^+(t), \eta_i^+(t)) \right]$; which can be written as

$$\begin{aligned}\nu &= (1 - \gamma)0 + \gamma(-\mu_i(t)), \\ &\text{for some } \gamma \in [0, 1],\end{aligned}\quad (35)$$

This implies that

$$\begin{aligned}&\frac{\partial^T H}{\partial x}(x) B_i F \left[\text{sat}(-\mu_i(t), -\eta_i^+(t), \eta_i^+(t)) \right] \\ &\quad + \frac{\partial^T H}{\partial x}(x) B_i \frac{\partial H_{c_i}}{\partial x_c}(x_c) \\ &= \left\{ -\gamma \frac{\partial^T H}{\partial x}(x) B_i B_i^T \frac{\partial H}{\partial x}(x) \right. \\ &\quad \left. + (1 - \gamma) \frac{\partial^T H}{\partial x}(x) B_i \frac{\partial H_{c_i}}{\partial x_c}(x_c) \mid \gamma \in [0, 1] \right\}\end{aligned}\quad (36)$$

Furthermore, when $\eta_i^+(t) < +\infty$, we have either

- $B_i^T \frac{\partial H}{\partial x}(x) \geq 0$ and $-B_i^T \frac{\partial H}{\partial x}(x) - \frac{\partial H_{c_i}}{\partial x_c}(x_c) > 0$ which implies $\frac{\partial^T H}{\partial x}(x) B_i \frac{\partial H_{c_i}}{\partial x_c}(x_c) \leq -\frac{\partial^T H}{\partial x}(x) B_i B_i^T \frac{\partial H}{\partial x}(x)$ or
- $B_i^T \frac{\partial H}{\partial x}(x) \leq 0$ and $-B_i^T \frac{\partial H}{\partial x}(x) - \frac{\partial H_{c_i}}{\partial x_c}(x_c) < 0$ which implies $\frac{\partial^T H}{\partial x}(x) B_i \frac{\partial H_{c_i}}{\partial x_c}(x_c) \leq -\frac{\partial^T H}{\partial x}(x) B_i B_i^T \frac{\partial H}{\partial x}(x)$ again.

So far, we can conclude that

$$\begin{aligned}&\frac{\partial^T H}{\partial x}(x) B_i \left(F \left[\text{sat}(-\mu_i(t), -\eta_i^+(t), \eta_i^+(t)) \right] + \frac{\partial H_{c_i}}{\partial x_c}(x_c) \right) \\ &\leq - \frac{\partial^T H}{\partial x}(x) B_i B_i^T \frac{\partial H}{\partial x}(x),\end{aligned}\quad (37)$$

i.e. $\max \tilde{\mathcal{L}}_F V_d(x, x_c) \leq -\frac{\partial^T H}{\partial x}(x) B B^T \frac{\partial H}{\partial x}(x)$.

By LaSalle's Invariance principle, the trajectories will converge to the largest invariant set, denoted as \mathcal{I} , within the set where $\{(x, x_c) \mid \dot{V} = 0\}$, i.e. $\{(x, x_c) \mid B^T \frac{\partial H}{\partial x}(x) = 0\}$. In \mathcal{I} we have

$$B^T \frac{\partial^2 H}{\partial x^2} B \text{sat} \left(-\frac{\partial H_c}{\partial x_c}(x_c(t)), -\eta^+(t), \eta^+(t) \right) = 0 \quad (38)$$

which implies that x remains at a constant value, denoted by ν , in \mathcal{I} and $\frac{\partial H}{\partial x}(\nu) = \alpha \mathbb{1}$. Furthermore, in view of $\nu \geq 0$ and the properties

of H and H_c we can prove that $\alpha > 0$. By the optimal control protocol given in the previous section, we have that $B \frac{\partial H_c}{\partial x_c}(x_c) = 0$. This concludes the proof.

Necessity. If the graph is not weakly connected then the above analysis will hold on every connected component, and the common value α will be different for different components. ■

Example 6.1 (Hydraulic network continued): In this example we show the simulation results of the hydraulic network defined on the graph given in Figure 3, with flow constraints given as solution of the optimization problem (21). The values of the parameters are taken as $S_i = 1m^2, i = 1, \dots, 5, \rho = 1kg/m^3, \gamma = 0$ and $[x(0), x_c(0)] = [0, 0.5, 1, 2, 0, 5, 9, 3, 0, -1, -2, -4]$. In Figure 4, it can be seen that the volume of each reservoir is kept nonnegative for all times. Furthermore the pressures of reservoirs converge to a common value (consensus).

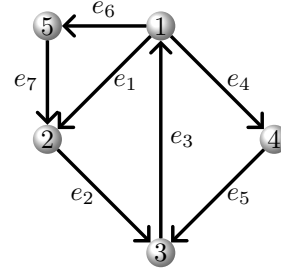


Fig. 3. Network structure of Example 6.1

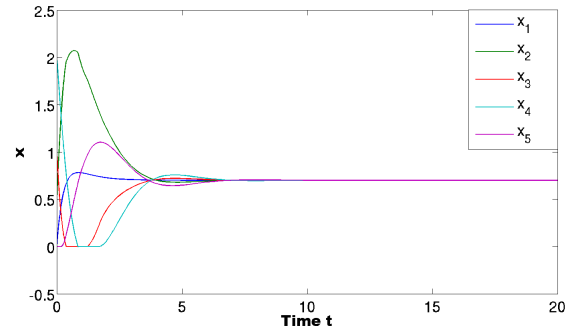


Fig. 4. The time-evolutions $x_1(t), x_2(t), x_3(t), x_4(t), x_5(t)$ of the system (23) defined on the graph as in Figure 3 using the solution of (21) as flow constraints.

VII. CONCLUSIONS

We have considered a basic model of dynamical distribution networks with state inequality constraints. We have formulated a distributed PI controller structure with time-varying flow constraints which achieves consensus and maintains the state constraints. The flow constraints have been expressed in terms of solutions of an optimization problem. We have discussed the existence of solutions for the system in the sense of Filippov, and carried out the stability analysis of the network by taking the Hamiltonian of the system as the Lyapunov function.

The results of this paper can be extended in a straightforward way to the case where the flows on the edges obey a priori constraints; for instance a limitation on the capacity of the pipes in hydraulic networks.